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KLEIN'S GROUP-THEORETIC CONCEPTION IN THE MECHANICS OF A MATERIAL POINT†

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The aim of this paper is to show that the physical principle of invariance and the condition that mechanics be Lagrangian make it possible to extend Klein's conception, originally proposed for geometry, to the mechanics of a material point. It thus becomes possible to place the known systems of mechanics (classical and relativistic) on a unified theoretical basis, after proving their uniqueness, and to outline a way to construct new, alternative systems.

This may prove useful for the interpretation of new experimental factors. It should be borne in mind that transformation groups are one of the most natural and universal tools for investigating and classifying fundamental phenomena in the exact sciences. As a mathematical object, groups can be studied theoretically in advance, independently of experimentation. Hence physical invariance principles, which develop Klein's conception, may be considered to be one of the simplest and most aesthetically justified sources of scientific predictions with a claim to reliability.

LOGICAL analysis of the schemes underlying the structure of classical and relativistic mechanics seems to imply that these systems are essentially distinct, lacking any unified basis. Nevertheless, all the necessary components of a unified conception of mechanics have long been available; they were prepared by the classicists of natural science.

Felix Klein formulated a special conception for geometry, which envisaged the construction of different geometries according to a general rule. Klein's idea was that every geometry is the theory of invariants of some transformation group [1]. The idea of axiomatizing an exact science, such as mechanics, was first proposed by Hilbert (Hilbert's Sixth Problem). He suggested using continuous groups for the purpose and studying all the mathematical alternatives thus created. To Einstein goes the credit for his principle of relativity, which generalizes Galileo's principle of relativity. Effectively, however, the group-theoretic world-view penetrated physics only after Poincaré created relativistic dynamics.

It was Poincaré who discovered the fundamental role of the Lorentz group in physics.

The group-theoretic approach has paved the way for important discoveries in theoretical physics. It was on that basis that Wigner [2] declared the priority of the principles of physical invariance over physical laws. Nevertheless, the idea of invariance as a justification of known models or a tool for setting up new models has not taken root.

1. STATEMENT OF THE PROBLEM

On the basis of Klein's group-theoretic conception, we wish to construct the mechanics of a material point (and its possible modifications): to define the basic concepts and see how to construct the law governing its motion.

The basic concepts are as follows:

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1. A reference system $[x, t]$ is a physical body (reference body) which has a clock at each point. The points of the body are arithmetized by variables x_1, x_2, x_3, which may be regarded as Cartesian coordinates. Time is arithmetized by a variable t.

Motions relative to the reference body of the system are considered with respect to a uniform time, established by synchronizing the readings of the clocks in the system.

2. Inertial reference systems are reference systems that transform into one another by transformations in a given group,

$$G_n\colon\ x' = \varphi(x, t, \tau), \qquad t' = \psi(x, t, \tau); \qquad [x', t'] \xleftrightarrow{G_n} [x, t]$$

and an isolated point at rest in any such system at some instant of time will remain at rest.

3. An inertial motion of a material point is a motion of an isolated material point in any inertial reference system.

Remark. Not every transformation group is suitable for constructing a meaningful material point. Even so, there are transformation groups that meet these demands and nevertheless define an inertial motion that is neither rectilinear nor uniform. This means that in such mechanical models Galileo's law of inertia fails to hold; that is, a "stationary" inertial observer will see moving inertial reference bodies as diffuse and blurred, rather than solid.

Let us assume that the following conditions hold: (a) the system of mechanics under construction is Lagrangian and the Lagrangian describes "pure" space–time; (b) the system is governed by the principle of invariance with respect to a given transformation group.

The first condition implies that the motions of a material point in any inertial reference system are described by the Lagrange equations; in particular, inertial motions are described by the Lagrange equations with zero on the right ("covariance"). The second condition implies that the equations of motion in any inertial reference system must have the same Lagrangian ("invariance").

After the Lagrangian has been found, its law of dynamics is represented by the Lagrange equations in which the right-hand sides are the projections of the force acting on the point. This automatically yields the transformation law for the force on passing from one inertial reference system to another.

Thus, the problem reduces to determining the Lagrangian.

2. THE RESULTS

Let us assume that an n-dimensional transformation group G_n, acting in four-dimensional real space $R^4 = \{x_1, x_2, x_3, t\}$, is defined by its Lie algebra A_n and that the covariance and invariance conditions hold. Then the most general Lagrangian L is defined by the equations

$$X_i^* L = (a_i - d\eta_i/dt)L + d\varphi_i(x, t)/dt, \qquad i = 1, \dots, n \tag{2.1}$$

$$X_i\varphi_j - X_j\varphi_i = \overset{e}{c_{ij}}\varphi_e + a_i\varphi_j - a_j\varphi_i + d_{ij} \tag{2.2}$$

$$\overset{e}{c_{ij}} d_{eh} + \overset{e}{c_{jh}} d_{ei} + \overset{e}{c_{hi}} d_{ej} = a_i d_{jh} + a_j d_{hi} + a_h d_{ij} \tag{2.3}$$

$$\overset{e}{c_{ij}} a_e = 0; \qquad e, j, h = 1, \dots, n. \tag{2.4}$$

Here the operators

$$X_i^* = X_i + \overset{i}{\zeta}_{x_\alpha^\cdot}(t, x, x^\cdot)\frac{\partial}{\partial x_\alpha^\cdot}, \qquad \alpha = 1, 2, 3$$

are extensions of the infinitesimal operators

$$X_i = \eta_i(t, x)\frac{\partial}{\partial t} + \overset{i}{\xi}_\alpha(t, x)\frac{\partial}{\partial x_\alpha}$$

to the velocity components $x_\alpha^\bullet$, computed by the standard formulas

$$\zeta^i_{\dot{x}_\alpha} = \frac{d\xi^i_\alpha}{dt} - \dot{x}_\alpha \frac{d\eta_i}{dt}$$

The set of operators X_i is a basis for the Lie algebra A_n of G_n:

$$A_n\colon\ X_1, \ldots, X_n \quad [X_i, X_j] = c^e_{ij} X_e$$

where φ_i are arbitrary solutions of Eqs (2.2) and a_i and d_{jh} are arbitrary constants satisfying Eqs (2.3) and (2.4). Our notation employs the repeated index summation convention.

3. SIMPLIFIED RESULT

Let

$$\Sigma_a = \sum_{i=1}^{n} a_i^2, \qquad \Sigma_\eta = \sum_{i=1}^{n} (d\eta_i/dt)^2$$

Proposition. Assume that:
1. The operators $X_1, \ldots, X_4$ form a basis of the ideal A_4 in the algebra

$$A_n\colon\ [X_\alpha, X_i] = c^\beta_{\alpha i} X_\beta; \qquad \alpha, \beta = 1, \ldots, 4.$$

2. If $c^p_{kl} . \delta_p = 0$, then $\delta_p = 0$; $k, l, p = 5, \ldots, n$.
3. The group G_4 that corresponds to A_4 is locally transitive: $\| \xi^A_\alpha\ , \eta_A \| = 4$.

Then, without loss of generality, the following Lagrangians will be valid for almost all $a \neq 0$:
If $\Sigma_a \neq 0$, $\Sigma_\eta \neq 0$, then $X_i^* L = (a_i - d\eta_i/dt) L$;
If $\Sigma_a \neq 0$, $\Sigma_\eta = 0$, then $L = 0$;
If $\Sigma_a = 0$, $\Sigma_\eta \neq 0$, then $X_i^* L = -(d\eta_i/dt) L + d\varphi_i^*/dt$;
If $\Sigma_a = 0$, $\Sigma_\eta = 0$, then $X_i^* L = d\varphi_i^*/dt$.
Here $\varphi_1^*, \ldots, \varphi_n^*$ denote some particular solution of Eq. (2.2).
Note that the conditions of the proposition are not necessary; neither is this the only possible simplification.

Examples: Galileo and Lorentz groups. These groups contain ten parameters each and satisfy the above conditions. They both contain the same subgroup G_7: the union of the group of motions of Euclidean space and the group of translations of the time t. The group G_4 whose Lie algebra is the ideal A_4 is a normal divisor of both groups, generated by the translations of coordinates and time.

The algebra of the Galileo group contains, besides the subalgebra A_7 of G_7, operators

$$X_8 = t\frac{\partial}{\partial x_1}, \qquad X_9 = t\frac{\partial}{\partial x_2}, \qquad X_{10} = t\frac{\partial}{\partial x_3}.$$

Equations (2.1)–(2.4) have a unique solution (apart from an unimportant additive constant), which corresponds to the classical expression for the kinetic energy: $L_G = 1/2mv^2$.

The algebra of the Lorentz group contains the following operators in addition to A_7 (c denotes the speed of light):

$$X_8 = t\frac{\partial}{\partial x_1} + \frac{x_1}{c^2}\frac{\partial}{\partial t}, \qquad X_9 = t\frac{\partial}{\partial x_2} + \frac{x_2}{c^2}\frac{\partial}{\partial t}, \qquad X_{10} = t\frac{\partial}{\partial x_3} + \frac{x_3}{c^2}\frac{\partial}{\partial t}$$

Equations (2.1)–(2.4) for the Lagrangians L_L, considered together with the limiting condition $\lim_{c\to\infty} L_L = L_G$, yield a solution which is again unique, apart from an unimportant additive constant—the Lagrangian for relativistic mechanics: $L_L = -mc^2\sqrt{1 - v^2/c^2}$.

Thus, the existence and uniqueness of classical and relativistic mechanics turn out to be a simple corollary of the assumption that the systems are Lagrangian and invariant with respect to the Galileo and Lorentz groups, respectively.

Remark. The application of invariance principles is naturally not limited to these examples. Meaningful mechanical models are obtained, for example, by considering conformal groups, groups of motions of de Sitter space and generalizations of the latter. The present author has in fact determined the inertial motions admitted by a conformal group [3, 4], the Lagrangian for the space associated with de Sitter space [5]:

$$L = -mc^2(1+\alpha t)^{-1}\sqrt{1-v^2/c^2}, \qquad \alpha = \text{const}$$

and the broadest possible generalization (in a well-defined sense) of the latter:

$$L = -mc^2(1+\alpha t+\beta_1 x_1+\beta_2 x_2+\beta_3 x_3)^{-1}\sqrt{1-v^2/c^2}, \qquad \beta_i = \text{const}$$

(this result is published here for the first time).

The admissible new groups need not be groups of motions of some four-dimensional Riemann space–time (like the Galileo group). In that case the methods of Riemannian geometry are useless, whereas the Klein approach is still valid.

4. PROOFS

Operator technique. The Lie algebra A_n and its extension to the velocity components have already been defined (Sec. 2). A second extension

$$X_i^{**} = X_i^{*} + \zeta_i(t, x, x^{\cdot}, L)\,\partial/\partial L$$

is an extension to a new variable L. The components ζ_i are not known in advance and will be determined later. But there is a known rule which, given the functions ζ_i, will construct a third extension

$$X_i^{***} = X_i^{**} + \overset{i}{\zeta}_{P_\epsilon}\partial/\partial P_\epsilon\,,\ \epsilon = 1, \dots, 7$$

to the partial derivatives of L with respect to t, x, $x^{\cdot}$. The functions $\zeta^i_{P_\epsilon}$ are determined by requiring that the differential form

$$dL = P_\alpha dx_\alpha + P_{\alpha+3}dx^{\cdot}_\alpha + P_4 dt$$

be invariant. We thus obtain an extension to the second partial derivatives.

An extension to the components of the acceleration of the material point is constructed in the same way.

Denote the operators, extended to all necessary variables, by X_i'.

The covariance conditions for the differential equations

$$\Phi_\alpha(t, x, x^{\cdot}, L, P, \dots) = 0 \tag{4.1}$$

are written as

$$X_i'\Phi_\alpha \,|_{\Phi_\alpha = 0} = 0 \tag{4.2}$$

We are treating Eqs (4.1) as algebraic equations in all the variables t, x, $x^{\cdot}$, L, P, ...

The invariance conditions (the Lagrangian must be the same in all inertial reference systems) may be written in the form

$$X_i^{**}(L - L(t, x, x^{\cdot})\,|_{L=L(t,x,x^{\cdot})} = 0 \tag{4.3}$$

Derivation of the formulas of Sec. 2. Our first task is to determine the functions ζ_i, on the basis of the assumption that the Lagrange equations are covariant.

To that end, following the rule, we write the latter as a system of algebraic equations

$$\Phi_\alpha \equiv P_{\alpha+3,4} + x^{\cdot}_\beta P_{\alpha+3,\beta} + x^{\cdot\cdot}_\beta P_{\alpha+3,\beta+3} - P_\alpha = 0; \qquad \alpha, \beta = 1, 2, 3$$

$$P_{\alpha+3,4} = \frac{\partial^2 L}{\partial x^{\cdot}_\alpha \partial t}, \qquad P_{\alpha+3,\beta} = \frac{\partial^2 L}{\partial x^{\cdot}_\alpha \partial x_\beta}, \qquad P_{\alpha+3,\beta+3} = \frac{\partial^2 L}{\partial x^{\cdot}_\alpha \partial x^{\cdot}_\beta}.$$

We then use the covariance conditions (4.2). Simplifying, we get

$$\frac{\partial^2 \zeta_i}{\partial L^2} = 0, \qquad \frac{\partial^2 \zeta_i}{\partial \dot{x}_\alpha \partial \dot{x}_\beta} = 0, \qquad \frac{\partial^2 \zeta_i}{\partial L \partial \dot{x}_\alpha} + \frac{\partial \eta_i}{\partial x_\alpha} = 0$$

$$\frac{\partial^2 \zeta_i}{\partial t\, \partial \dot{x}_\alpha} + \dot{x}_\beta \frac{\partial^2 \zeta_i}{\partial \dot{x}_\alpha \partial \dot{x}_\beta} - \frac{\partial \zeta_i}{\partial x_\alpha} = 0$$

$$\frac{\partial^2 \zeta_i}{\partial L\, \partial t} + \dot{x}_\alpha \frac{\partial^2 \zeta_i}{\partial L\, \partial x_\alpha} + \frac{\partial^2 \eta_i}{\partial t^2} + 2\dot{x}_\alpha \frac{\partial^2 \eta_i}{\partial x_\alpha \partial t} + \dot{x}_\alpha \dot{x}_\beta \frac{\partial^2 \eta_i}{\partial x_\alpha \partial x_\beta} = 0$$

Integrating we obtain

$$\zeta_i(t, x, \dot{x};\ L) = \left(a_i - \frac{d\eta_i}{dt}\right) L + \frac{d\varphi_i(t, x)}{dt} \tag{4.4}$$

where a_i are arbitrary constants and φ_i are arbitrary differentiable functions.

The commutation relations $[X_i^{**}, X_j^{**}] = c_{ij}^e X_e^{**}$ yield the equations

$$X_i^* \frac{d\eta_j}{dt} - X_j^* \frac{d\eta_i}{dt} - c_{ij}^e \left(a_e - \frac{d\eta_e}{dt}\right) = 0$$

$$X_i^* \frac{d\varphi_j}{dt} - X_j^* \frac{d\varphi_i}{dt} - c_{ij}^e \frac{d\varphi_e}{dt} - \left(a_i - \frac{d\eta_i}{dt}\right) \frac{d\varphi_j}{dt} + \left(a_j - \frac{d\eta_j}{dt}\right) \frac{d\varphi_i}{dt} = 0$$

Using the easily verified commutation relation

$$X_i^* \frac{d\psi(t, x)}{dt} - \frac{d}{dt} X_i \psi(t, x) = -\frac{d\eta_i}{dt} \frac{d\psi(t, x)}{dt}$$

which is true for any function $\psi(t, x)$, we obtain Eqs (2.4) and (2.2). The constants of integration d_{ij} must be such that system (2.2) is complete; they may be found in the form

$$X_i(X_j\varphi_h - X_h\varphi_j) + X_j(X_h\varphi_i - X_i\varphi_h) + X_h(X_i\varphi_j - X_j\varphi_i) = X_i\chi_{jh} + X_j\chi_{hi} + X_h\chi_{ij}$$

where χ_{ij} are the right-hand sides of Eqs (2.2). Simplifying, we obtain Eqs (2.3). Finally, the invariance conditions (4.3), together with (4.4), yield Eqs (2.1).

Proof of the proposition in Sec. 3. We shall first prove that, under the assumptions of the proposition, the general solution of equations (2.2) may be written in the form

$$\varphi_i = \varphi_i^* + X_i\nu_1 - a_i\nu_1, \qquad i = 1, \dots, n \tag{4.5}$$

where $\varphi_1^*, \dots, \varphi_n^*$ is some particular solution and ν_1 is an arbitrary differentiable function. The proof of (4.5) is divided into seven simple steps.

1. It follows from (2.4) and conditions 1 and 3 of the proposition that there exists a function μ such that $X_\alpha \mu = a_\alpha$, $\alpha = 1, \dots, 4$.

2. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be the first four functions of an arbitrary fixed solution of Eqs (2.2), corresponding to the ideal A_4 of A_n. The existence of μ then implies the existence of a function ν such that $X_\alpha \nu = (\varphi_\alpha - \varphi_\alpha^*)e^{-\mu}$.

3. Replace the remaining functions $\varphi_5, \dots, \varphi_n$ of the fixed solution by new variables $\gamma_5, \dots, \gamma_n$, using the formulas $\varphi_k = \varphi_k^*(X_k\nu + \gamma_k)$; $k = 5, \dots, n$.

The equations for φ_k yield equations for γ_k: $X_\alpha \gamma_k = (X_k\mu - a_k)X_\alpha \nu$.

4. Put $a_k^* = X_k\mu$. Since μ exists, the system of n equations in which it occurs is complete. This yields the commutation relations $X_\alpha a_k^* = c_{\alpha k}^\beta a_\beta = 0$, which may be satisfied only by constants $a_k^* = \text{const}$.

5. Condition 2 and the fact that a_k^* are constants imply $a_k^* = a_k$.

But then the equations for γ_k and condition 3 imply that $\gamma_k = \text{const}$.

6. The quantities γ_k satisfy the equations $X_k\gamma_l - X_l\gamma_k = c_{kl}^p\gamma_p$; $k, l, p = 5, \dots, n$. Since $\gamma_k = \text{const}$, condition 2 implies $\gamma_k = 0$.

7. We have thus proved that all the functions φ_1, which by assumption form a given solution of Eqs (2.2), can be written uniformly as

$$\varphi_i = \varphi_i^* + e^\mu X_i\nu = \varphi_i^* + X_i\nu_1 - a_i\nu_1, \qquad \nu_1 = \nu e^\mu$$

This proves formulas (4.5). They may now be used to reduce Eqs (2.1) to the form

$$X_i^*\left(L-\frac{dv_1}{dt}\right)=\left(a_i-\frac{d\eta_i}{dt}\right)\left(L-\frac{dv_1}{dt}\right)+\frac{d\varphi_i^*}{dt}$$

Hence it follows that the general solution $\varphi_1, \ldots, \varphi_n$ in (2.1) may be replaced, without loss of generality, by the particular solution $\varphi_1^*, \ldots, \varphi_n^*$.

We shall now show that, if not all the constants a vanish, we may assume that all the functions $\varphi_1^*, \ldots, \varphi_n^*$ are constants.

Denote these presumed constants by $b_1, \ldots, b_n$. Let $a_1 \neq 0$. It then follows from Eqs (2.2) that

$$\chi_{1i}^0 \equiv \overset{e}{c}_{1i} b_e + a_1 b_i - a_i b_1 + d_{1i} = 0; \quad i, e = 2, \ldots, n$$

If $b_1 = 0$, these $n-1$ equations may be used to determine all the other numbers b_i. Indeed, if $d_{1i} = 0$, we can take $b_i = 0$. If $d_{1i} \neq 0$, we can find b_i for all the real values of a_1 for which the determinant of the system does not vanish. Obviously, the number of values of a_1 for which this is impossible is at most $n-1$. Thus, for almost all a_1, we have $\chi_{1i}^0 = 0$.

We shall now show that the values of b_i thus determined make the remaining quantities $\chi_{ij}^0 = \chi_{ij}|_{\varphi_e = b_e}$ vanish, so that they satisfy the remaining equations (2.2). We shall use Eqs (2.3), replacing d_{ij} in them by χ_{ij}^0 via the formulas $d_{ij} = \chi_{ij}^0 - \overset{e}{c}_{ij}$, $b_e - a_i b_j + a_j b_i$. We claim that the numbers χ_{ij}^0 satisfy the same equations as d_{ij}:

$$\overset{e}{c}_{ij}\chi_{eh}^0 + \overset{e}{c}_{jh}\chi_{ei}^0 + \overset{e}{c}_{hi}\chi_{ej}^0 = a_i\chi_{jh}^0 + a_j\chi_{hi}^0 + a_h\chi_{ij}^0$$

Setting $e = 1$ in these equations, we get

$$\overset{e}{c}_{j1}\chi_{ei}^0 + \overset{e}{c}_{1i}\chi_{ej}^0 = a_1\chi_{ij}^0 \tag{4.6}$$

This is a system of homogeneous linear equations in the unknowns χ_{ij}^0 $(i, j \neq 1)$. There are clearly as many equations in this system as there are unknowns. For those real values of $a_1 \neq 0$ for which the determinant of system (4.6) does not vanish, we have $\chi_{ij}^0 = 0$. At most $(n-1)(n-2)/2$ values of a_1 may fail to satisfy this condition.

Examples. Considering the group of motions of Euclidean space and the group of time translations; which are subgroups of both Galileo and Lorentz groups, we conclude that the Lagrangian in both cases depends only on the velocity: $L = f(v^2)$, $v^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$.

Galileo group. In the non-trivial case, when $a_i = 0$, the last three equations of system (2.1) become

$$\frac{\partial L_G}{\partial \dot{x}_1} = a_{8\beta}\dot{x}_\beta + a_{80}, \qquad \frac{\partial L_G}{\partial \dot{x}_2} = a_{9\beta}\dot{x}_\beta + a_{90}, \qquad \frac{\partial L_G}{\partial \dot{x}_3} = a_{10,\beta}\dot{x}_\beta + a_{10,0}.$$

Since the Lagrangian L depends only on the velocity, these equations reduce to a single equation

$$2f'(v^2) = a_{81} = a_{92} = a_{10,3} = m = \text{const}$$

Hence we obtain $L_G = \frac{1}{2}mv^2 + k$. The additive constant k is not essential.

Lorentz group. The structure constants of the group are such that necessarily $a_1 = \ldots = a_{10} = 0$. The last three equations of the system become

$$X_8^* L_L = -\dot{x}_1 L_L + \frac{d\varphi_8}{dt}, \quad X_9^* L_L = -\dot{x}_2 L_L + \frac{d\varphi_9}{dt}, \quad X_{10}^* L_L = -\dot{x}_3 L_L + \frac{d\varphi_{10}}{dt}.$$

Since the Lagrangian L_L depends only on the velocity, these equations reduce to a single equation: $2(c^2 - v^2)f'(v^2) + f(v^2) = b'$, $b' = \text{const}$. The solution $L_L = b_0\sqrt{1 - v^2/c^2} + b'$, which also satisfies the condition $\lim_{c\to\infty} L_L = L_G$, is exactly (apart from an additive constant) the same as the Lagrangian L_L of relativistic mechanics: $L_L = -mc^2\sqrt{1 - v^2/c^2}$.

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REFERENCES

1. KLEIN F., *Vergleichende Betrachtungen über neuere geometrische Forschungen (Erlangen-Programm)*. Verlag A. Deichert, Erlangen, 1872.
2. WIGNER E., Symmetry and conservation laws. *Physics Today* **17**, 34–40, 1964.
3. MARKHASHOV L. M., On conformally invariant motions of a material point. *Prikl. Mat. Mekh.* **30**, 3–13, 1966.
4. MARKHASHOV L. M., On relativistic analogues of the dynamics of a material point. *Prikl. Mat. Mekh.* **53**, 382–395, 1989.
5. MARKHASHOV L. M., On a possible alternative in relativistic mechanics. In *Proceedings of the Fifth All-Union Conference on Analytical Mechanics, Theory of Stability and Control of Motion*. Vychisl. Tsentr, Akad. Nauk SSSR, Moscow, 1990.

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REVERSIBLE SYSTEMS. STABILITY AT 1:1 RESONANCE†

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Some general properties of reversible systems are studied: the nature of the stability of the trivial equilibrium position, the conditions for the existence of certain periodic solutions and symmetry of the phase portrait. It is shown that a discrete automorphism (symmetry) group generates integral manifolds. A detailed investigation of the stability of the trivial solution at 1:1 resonance is presented. The necessary and sufficient conditions for the stability of a model system are obtained and it is shown that instability of the system implies instability of the complete system.

1. SOME PROPERTIES OF REVERSIBLE SYSTEMS

CONSIDER an autonomous system of differential equations

$$dx_s/dt = f_s(x_1, \ldots, x_n) \quad (s = 1, 2, \ldots, n) \tag{1.1}$$

with smooth right-hand sides, whose phase flow is reversible [1, p 115]: there exists a non-degenerate linear mapping

$$\mathbf{M}: \ \mathbf{X} \to \mathbf{X}, \ \ t \to -t \tag{1.2}$$

such that

$$\mathbf{f}(\mathbf{x}) = -\mathbf{M}^{-1}\mathbf{f}(\mathbf{M}\mathbf{x}) \tag{1.3}$$

$$\mathbf{f} = (f_1, \ldots, f_n), \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$$

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